

Putting a cap on causality violations in causal dynamical triangulations

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ABSTRACT: The formalism of causal dynamical triangulations (CDT) provides us with a non-perturbatively defined model of quantum gravity, where the sum over histories includes only causal space-time histories. Path integrals of CDT and their continuum limits have been studied in two, three and four dimensions. Here we investigate a generalization of the two-dimensional CDT model, where the causality constraint is partially lifted by introducing branching points with a weight g_s , and demonstrate that the system can be solved analytically in the genus-zero sector. The solution is analytic in a neighborhood around weight $g_s = 0$ and cannot be analytically continued to $g_s = \infty$, where the branching is entirely geometric and where one would formally recover standard Euclidean two-dimensional quantum gravity defined via dynamical triangulations or Liouville theory.

KEYWORDS: Lattice Models of Gravity, 2D Gravity, Models of Quantum Gravity.

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1. Introduction

The idea of CDT, by which we mean the definition of quantum gravity theory via causal dynamical triangulations, is two-fold: firstly, inspired by earlier ideas in the continuum theory [1, 2], we insist, starting from space-times with a Lorentzian signature, that only causal histories contribute to the quantum gravitational path integral. Secondly, we assume the presence of a global time-foliation.

The formalism of dynamical triangulations (DT) provides a simple regularization of the sum over *geometries* by providing a grid of piecewise linear geometries constructed from elementary building blocks (these are d -dimensional simplices of identical size and shape if we want to construct a d -dimensional geometry, see [3, 4] for reviews). The ultraviolet cut-off is given by the edge length of the building blocks. The causal variant CDT also uses DT as the regularization of the path integral. A detailed description of which causal geometries are included in the grid can be found in references [5, 6].

We emphasize that the use of triangulations is merely a technical regularization of the assumed underlying continuum theory, in the same way a lattice can be used for regularizing a quantum field theory. By no means do we presuppose that space-time is literally made out of little simplices. Some support for the existence of an underlying (non-perturbative) continuum quantum field theory in higher dimensions has been provided in [6]–[11] and seems to be in qualitative agreement with independent analyses carried out using the renormalization group [12–16].

While the CDT model is defined as a sum over causal space-time histories (each with an appropriate weight), one can ask whether this causality constraint can be lifted. One motivation for introducing it was that unrestricted summation over space-time histories¹ leads to a dominance of highly singular configurations in dimensions $d > 2$, which prevents the existence of a physically meaningful continuum limit of the regularized lattice

¹Even when keeping the space-time topology fixed

theory [17–19]. On the other hand, if one takes the point of view that a maximal number of possible fluctuations should be included in the path integral (while still leading to a meaningful result), one may wonder whether it is possible to reintroduce (a subclass of) configurations into the sum over geometries which correspond to metric structures with causality violations. The question we would like to pose is whether this can be done in a controlled manner — using the additional time-slicing structure present in CDT — which avoids the problems encountered previously by DT, corresponding to an unrestricted inclusion of all such configurations.

Because of the ready availability of analytic tools and the existence of analytical solutions, we will in a first step analyze the situation in two dimensions. In this context, the issue has been addressed previously in a 2d toy model, focussing on the effects of including a class of minimal wormholes in CDT, which can be said to violate causality only mildly and are much less abundant than general wormholes [20–22]. In the present work, we will look at the genus-zero sector of a generalized model of two-dimensional CDT, which in principle allows for the inclusion of arbitrary space-time topologies, as well as “outgrowths”, that is, the sprouting of baby universes, and associates with them a weight depending on the gravitational coupling constant.² A key observation is that requiring the propagator of the model to reduce to that of standard CDT when the bare coupling is taken to zero uniquely fixes the scaling of this coupling, leading to a continuum limit where branching processes occur, but are scarce compared to the situation in DT.

The following sections deal with explaining this scaling argument, and with analytically computing the genus-0 propagator (or loop-loop amplitude) and corresponding disc amplitude. We find a disc amplitude which is analytic around the rescaled coupling constant $g_s = 0$ (see eq. (2.14) below for the precise definition of g_s). However, the radius of convergence is finite and the amplitude becomes complex if one increases g_s beyond this radius of convergence. In the current framework it is therefore impossible to perform an interpolation to $g_s = \infty$, which is where one would formally make contact with ordinary DT and Liouville theory.

The computation of higher-genus amplitudes in this framework is currently under way, and may open new perspectives on the issue of the sum over topologies in theories of quantum gravity, which are only apparent in a formulation that has at least some memory of the *Lorentzian* structure of space-time built in. However, we have as yet no definite statements to make about the properties of general higher-genus amplitudes, the summability of the genus expansion or a generalization of the model to higher dimensions.

2. Two-dimensional CDT and euclidean quantum gravity

Two-dimensional quantum gravity is a wonderful playground for “quantum geometry”, understood as the statistical sum over geometries. The reason for this is that the action is trivial as long as we ignore topology changes (and even then it is almost trivial). One can therefore use entirely geometric reasoning to derive relations between or properties of

²The first analysis of a CDT model with local “decorations” was made in [23].

“Green’s functions”.³ In this context it is convenient to study the proper-time “propagator”, namely, the amplitude of geometries with two space-like boundaries separated by a proper time (or geodesic distance) t . Although the proper-time propagator is a special amplitude, it has the virtue that other amplitudes, like the disc or cylinder amplitudes, can be calculated from it [29–32, 5]. When the path integral representation of this propagator is defined in the Lorentzian domain, using CDT, we can associate with each of the causal, piecewise linear Lorentzian space-time geometries a unique Euclidean geometry. After this rotation we perform the sum over the Euclidean geometries thus obtained. The sum is now different from the usual Euclidean sum over geometries, since it extends only over a strict subset of all Euclidean configurations, leading to an alternative quantization of 2d quantum gravity (CDT). In the end, we can rotate back the propagator from Euclidean to Lorentzian proper time if needed. In the remainder of this article we will stay in the Euclidean regime, as defined above.

For ease of presentation, we will in the following use a continuum notation. A derivation of the continuum expressions from the regularized (lattice) expressions can be found in [5]. We will assume that space-time has the topology $S^1 \times [0, 1]$. After rotation to Euclidean signature, the action is

$$S[g_{\mu\nu}] = \lambda \int d^2\xi \sqrt{\det g_{\mu\nu}(\xi)} + x \oint dl_1 + y \oint dl_2, \tag{2.1}$$

where λ is the cosmological constant, x and y are two so-called boundary cosmological constants, $g_{\mu\nu}$ is the metric of a geometry of the kind described above, and the line integrals refer to the lengths of the in- and out-boundaries induced by $g_{\mu\nu}$. The propagator $G_\lambda(x, y; t)$ is defined by

$$G_\lambda(x, y; t) = \int \mathcal{D}[g_{\mu\nu}] e^{-S[g_{\mu\nu}]}, \tag{2.2}$$

where the functional integration is over all *causal* geometries $[g_{\mu\nu}]$ such that the final boundary with boundary cosmological constant y is separated a geodesic distance t from the initial boundary with boundary cosmological constant x .⁴ Calculating the path integral (2.2) with the help of the CDT regularization and taking the continuum limit as the side-length a of the simplices goes to zero leads to the equation [5]

$$\frac{\partial}{\partial t} G_\lambda(x, y; t) = -\frac{\partial}{\partial x} \left[(x^2 - \lambda) G_\lambda(x, y; t) \right], \tag{2.3}$$

which is solved by⁵

$$G_\lambda(x, y; t) = \frac{\bar{x}^2(t, x) - \lambda}{x^2 - \lambda} \frac{1}{\bar{x}(t, x) + y}, \tag{2.4}$$

³An early example is the proof [24] that the string tension of bosonic string theory (regularized using DT [25, 26]) does not scale, thus providing a simple geometric understanding of the impossibility of defining bosonic string theory in target space dimensions larger than or equal to 2. Other applications in non-critical string theory can be found in [27, 28].

⁴This class of geometries is difficult to define directly in a continuum, gauge-fixed formulation; what we have in mind here is the continuum limit of the corresponding discrete sum.

⁵The asymmetry between x and y is due to the convention that the initial boundary contains a marked point. Symmetric expressions where neither or both boundaries have marked points can be found in [33].

where $\bar{x}(t, x)$ denotes the solution of the characteristic equation for (2.3), namely,

$$\frac{d\bar{x}}{dt} = -(\bar{x}^2 - \lambda), \quad \bar{x}(0, x) = x. \quad (2.5)$$

Let l_1 denote the length of the initial and l_2 the length of the final boundary. Rather than considering a situation where the boundary cosmological constant x is fixed, we will take l_1 as fixed, and denote the corresponding propagator by $G_\lambda(l_1, y; t)$, with similar definitions for $G_\lambda(x, l_2; t)$ and $G_\lambda(l_1, l_2; t)$. All of them are related by Laplace transformations, for instance,

$$G_\lambda(x, y; t) = \int_0^\infty dl_2 \int_0^\infty dl_1 G_\lambda(l_1, l_2; t) e^{-xl_1 - yl_2}, \quad (2.6)$$

where the Laplace-transformed propagator obeys the composition rule

$$G_\lambda(x, y; t_1 + t_2) = \int_0^\infty dl G_\lambda(x, l; t_1) G_\lambda(l, y; t_2). \quad (2.7)$$

Eq. (2.7) is the simplest example of the use of quantum geometry. While the property (2.7) is evident in the context of CDT where no baby universes are allowed, it is also true in Euclidean quantum gravity (where there is no such constraint), if one defines the distance between the initial and final loop appropriately [29, 34].

Another, slightly more complicated example is illustrated graphically by figure 1, which implies the functional relation

$$-\frac{\partial W_\lambda(x)}{\partial \lambda} = \int_0^\infty dt \int_0^\infty dl G_\lambda(x, l; t) l W_\lambda(l). \quad (2.8)$$

It encodes the following: let $W_\lambda(l)$ denote the disc amplitude, i.e. the Hartle-Hawking amplitude with a fixed boundary length l , and $W_\lambda(x)$ the corresponding Laplace-transformed amplitude where x is a fixed boundary cosmological constant. Differentiation with respect to the cosmological constant λ means marking a point in the bulk, as shown in the figure. Each configuration appearing in the path integral has a unique decomposition into a cylinder of proper-time extension t , (where the proper time is defined as the geodesic distance of the marked point to the boundary), and the disc amplitude itself, as summarized in eq. (2.8).

Starting from a regularized theory with a cut-off a , it was shown in [5] that there are two natural solutions to eq. (2.8). In one of them, the regularized disc amplitude diverges with the cut-off a and the geodesic distance t scales canonically with the lattice spacing a according to

$$W_{\text{reg}} \xrightarrow{a \rightarrow 0} a^\eta W_\lambda(x), \quad \eta < 0, \quad (2.9)$$

$$t_{\text{reg}} \xrightarrow{a \rightarrow 0} t/a^\varepsilon, \quad \varepsilon = 1. \quad (2.10)$$

In the other, the scaling goes as

$$W_{\text{reg}} \xrightarrow{a \rightarrow 0} \text{const.} + a^\eta W_\lambda(x), \quad \eta = 3/2 \quad (2.11)$$

$$t_{\text{reg}} \xrightarrow{a \rightarrow 0} t/a^\varepsilon, \quad \varepsilon = 1/2, \quad (2.12)$$

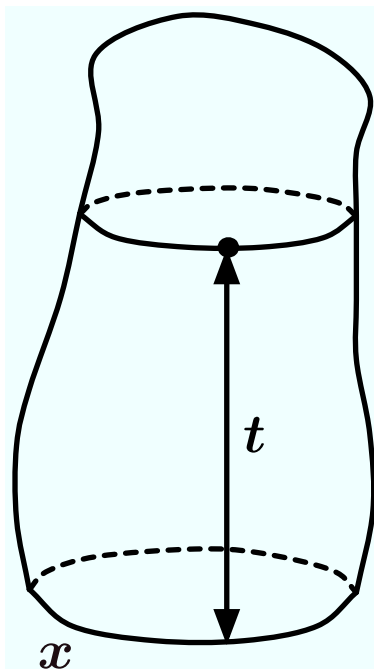


Figure 1: Graphical representation of relation 2.8: differentiating the disc amplitude $W_\lambda(x)$ (represented by the entire figure) with respect to the cosmological constant λ corresponds to marking a point somewhere inside the disc. This point has a geodesic distance t from the initial loop. Associated with the point one can identify a connected curve of length l , all of whose points also have a geodesic distance t to the initial loop. This loop can now be thought of as the curve along which the lower part of the figure (corresponding to the loop-loop propagator $G_\lambda(x, l; t)$) is glued to the cap, which itself is the disc amplitude $W_\lambda(l)$.

where the subscript “*reg*” denotes the regularized quantities in the discrete lattice formulation. The first scaling (2.9)-(2.10), with $\eta = -1$, is encountered in CDT, while the second scaling (2.11)-(2.12) is realized in Euclidean gravity, i.e. Liouville gravity or gravity defined from matrix models.

As demonstrated in [5], it is possible to treat both models simultaneously. Allowing for the creation of baby universes during the “evolution” in proper time t (by construction, a process forbidden in CDT) leads to a generalization of (2.3), namely,

$$a^\varepsilon \frac{\partial}{\partial t} G_{\lambda,g}(x, y; t) = -\frac{\partial}{\partial x} \left[\left(a(x^2 - \lambda) + 2g a^{\eta-1} W_{\lambda,g}(x) \right) G_{\lambda,g}(x, y; t) \right], \quad (2.13)$$

where we have introduced a new coupling constant g , associated with the creation of baby universes, and also made the additional dependence explicit in the amplitudes. In [5] it was noted that for $g = 1$, that is, viewing this creation as a purely geometric process,⁶ one obtains Euclidean quantum gravity. This happens because according to (2.9) and (2.11), we have either $\eta = -1$, which is inconsistent with (2.13), or we have from (2.11) that

⁶By this we mean that each distinct geometry (distinct in the sense of Euclidean geometry) appears with equal weight in the sum over two-dimensional geometries.

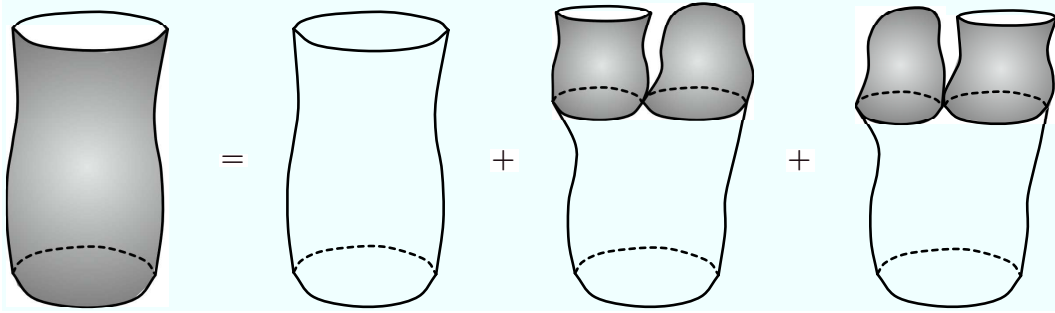


Figure 2: In all four graphs, the geodesic distance from the final to the initial loop is given by t . Differentiating with respect to t leads to eq. (2.15). Shaded parts of graphs represent the full, g_s -dependent propagator and disc amplitude, and non-shaded parts the CDT propagator.

$\eta = 3/2$ and thus $\varepsilon = 1/2$, which is consistent with (2.11). On the other hand, setting $g = 0$, thereby forbidding the creation of baby universes, leads of course back to (2.3).

In a non-trivial extension of previous work, we will now allow for the possibility that the coupling g becomes a non-constant function $g = g(a)$ of the cut-off a . A geometric interpretation of this assignment will be given in the discussion section below. Since we are interested in a theory which smoothly recovers CDT in the limit as $g \rightarrow 0$, it is natural to assume that $\eta = -1$, like in CDT. Consequently, the only way to obtain a non-trivial consistent equation is to assume that g scales to zero with the cut-off a according to

$$g = g_s a^3, \quad (2.14)$$

where g_s is a coupling constant of mass dimension three, which is kept constant when $a \rightarrow 0$. With this choice, eq. (2.13) is turned into

$$\frac{\partial}{\partial t} G_{\lambda, g_s}(x, y; t) = -\frac{\partial}{\partial x} \left[\left((x^2 - \lambda) + 2g_s W_{\lambda, g_s}(x) \right) G_{\lambda, g_s}(x, y; t) \right]. \quad (2.15)$$

The graphical representation of eq. (2.15) (or (2.13) for $g \neq 0$) is shown in figure 2. Differentiating the integral equation corresponding to this figure with respect to the time t obtains (2.15). The disc amplitude $W_{\lambda, g_s}(x)$ is at this stage unknown.

Note that one could in principle have considered an a priori more general branching process, where more than one baby universe is allowed to sprout at any given time step t . However, one observes from the scaling relation (2.14) that the corresponding extra terms in relation (2.13) would be suppressed by higher orders of a and therefore play no role in the continuum limit.

In the next section we will show that quantum geometry, in the sense defined above, together with the requirement of recovering standard CDT in the limit as $g_s \rightarrow 0$, uniquely determines the disc amplitude and thus $G_{\lambda, g_s}(x, y; t)$.

3. The disc amplitude

The disc amplitude of CDT was calculated in [5, 35]. In [5] it was determined directly by integrating $G_\lambda(l_1, l_2 = 0; t)$ over all times. This decomposition is unique, since by

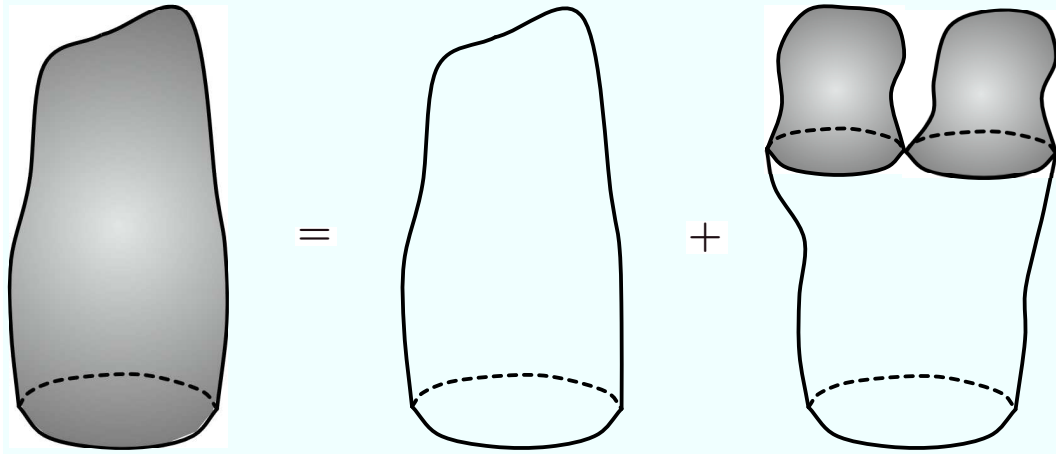


Figure 3: Graphical illustration of eq. (3.2). Shaded parts represent the full disc amplitude, unshaded parts the CDT disc amplitude and the CDT propagator.

assumption t is a global time and no baby universes can be created. In [35] it was shown that it could also be obtained from Euclidean quantum gravity (matrix model results) by peeling off baby universes in a systematic way. By either method one finds

$$W_\lambda(x) = \frac{1}{x + \sqrt{\lambda}} \quad (3.1)$$

for the disc amplitude as function of the boundary cosmological constant x . In the present, generalized case we allow for baby universes, leading to a graphical representation of the decomposition of the disc amplitude as shown in figure 3. It translates into the equation

$$W_{\lambda, g_s}(x) = W_{\lambda, g_s}^{(0)}(x) + g_s \int_0^\infty dt \int_0^\infty dl_1 dl_2 (l_1 + l_2) G_{\lambda, g_s}^{(0)}(x, l_1 + l_2; t) W_{\lambda, g_s}(l_1) W_{\lambda, g_s}(l_2) \quad (3.2)$$

for the full propagator $W_{\lambda, g_s}(x)$, where we have introduced a superscript (0) to indicate the CDT amplitudes, that is,

$$W_{\lambda, g_s}^{(0)}(x) \equiv W_{\lambda, g_s=0}(x) = W_\lambda(x), \quad (3.3)$$

and similarly for $G_{\lambda, g_s}^{(0)}$, quantities which were defined in eqs. (3.1) and (2.4) respectively. The integrations in (3.2) can be performed, yielding

$$W_{\lambda, g_s}(x) = \frac{1}{x + \sqrt{\lambda}} + \frac{g_s}{x^2 - \lambda} \left(W_{\lambda, g_s}^2(\sqrt{\lambda}) - W_{\lambda, g_s}^2(x) \right). \quad (3.4)$$

Solving for $W_{\lambda, g_s}(x)$ we find

$$W_{\lambda, g_s}(x) = \frac{-(x^2 - \lambda) + \hat{W}_{\lambda, g_s}(x)}{2g_s}, \quad (3.5)$$

where we have defined

$$\hat{W}_{\lambda,g_s}(x) = \sqrt{(x^2 - \lambda)^2 + 4g_s \left(g_s W_{\lambda,g_s}^2(\sqrt{\lambda}) + x - \sqrt{\lambda} \right)}. \quad (3.6)$$

The sign of the square root is fixed by the requirement that $W_{\lambda,g_s}(x) \rightarrow W_\lambda(x)$ for $g_s \rightarrow 0$, and $W_{\lambda,g_s}(x)$ is determined up to the value $W_{\lambda,g_s}(\sqrt{\lambda})$. We will now show that this value is also determined by consistency requirements of the quantum geometry. If we insert the solution (3.5) into eq. (2.15) we obtain

$$\frac{\partial}{\partial t} G_{\lambda,g_s}(x, y; t) = -\frac{\partial}{\partial x} \left[\hat{W}_{\lambda,g_s}(x) G_{\lambda,g_s}(x, y; t) \right]. \quad (3.7)$$

In analogy with (2.4) and (2.5), this is solved by

$$G_{\lambda,g_s}(x, y; t) = \frac{\hat{W}_{\lambda,g_s}(\bar{x}(t, x))}{\hat{W}_{\lambda,g_s}(x)} \frac{1}{\bar{x}(t, x) + y}, \quad (3.8)$$

where $\bar{x}(t, x)$ is the solution of the characteristic equation for (3.7),

$$\frac{d\bar{x}}{dt} = -\hat{W}_{\lambda,g_s}(\bar{x}), \quad \bar{x}(0, x) = x, \quad (3.9)$$

such that

$$t = \int_{\bar{x}(t)}^x \frac{dy}{\hat{W}_{\lambda,g_s}(y)}. \quad (3.10)$$

Physically, we require that t can take values from 0 to ∞ , as opposed to just in a finite interval. From expression (3.10) for t this is only possible if the polynomial under the square root in the defining equation (3.6) has a double zero, which fixes the function $\hat{W}_{\lambda,g_s}(x)$ to

$$\hat{W}_{\lambda,g_s}(x) = (x - \alpha) \sqrt{(x + \alpha)^2 - 2g_s/\alpha}, \quad (3.11)$$

where

$$\alpha = u\sqrt{\lambda}, \quad u^3 - u + \frac{g_s}{\lambda^{3/2}} = 0. \quad (3.12)$$

In order to have a physically acceptable $W_{\lambda,g_s}(x)$, one has to choose the solution to the third-order equation which is closest to 1. Quite remarkably, one can also derive (3.11) from (3.5) by demanding that the inverse Laplace transform $W_{\lambda,g_s}(l)$ fall off exponentially for large l . In this region $W_{\lambda,g_s}(x)$ equals $W_{\lambda,g_s}^{(0)}(x)$ plus a convergent power series in the dimensionless coupling constant $g_s/\lambda^{3/2}$. It can be seen from eq. (3.12) and (3.11) that the solution (3.11) is analytic in a neighborhood of $g_s = 0$. However, for sufficient large g_s (3.11) will be complex and therefore cannot be continued to the (real) disc amplitude known from DT or Liouville theory. Formally this theory corresponds to $g = 1$, i.e. $g_s = \infty$, see (2.14).

One can check the consistency of the quantum geometry by noting that using (3.8) in (2.8) the integration can be performed to yield

$$\frac{\partial W_{\lambda,g_s}(x)}{\partial \lambda} = \frac{W_{\lambda,g_s}(x) - W_{\lambda,g_s}(\alpha)}{\hat{W}_{\lambda,g_s}(x)}, \quad (3.13)$$

which is indeed satisfied by the solution (3.5).

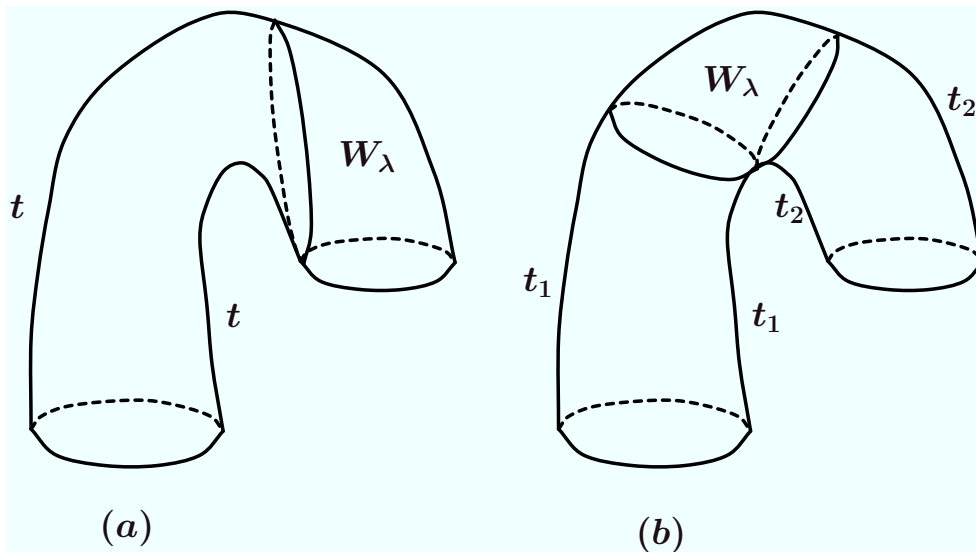


Figure 4: Two different ways of decomposing the loop-loop amplitude into proper-time propagators and a disc amplitude. Two points touch in the disc amplitude W , pinching the boundary to a figure-8, which combinatorially implies a substitution $W_{\lambda, g_s}(l) \rightarrow lW_{\lambda, g_s}(l)$ in the formulas. The time variables are related by $t_1 + t_2 = t$.

4. The loop-loop amplitude

We mentioned above that the loop-loop propagator can be regarded as a building block for other, more conventional “observables” in 2d quantum gravity. One of the most beautiful illustrations of this and at the same time a non-trivial example of what we have called quantum geometry is the calculation in 2d Euclidean quantum gravity of the loop-loop amplitude from the loop-loop proper-time propagator [30]. The full loop-loop amplitude is obtained by summing over all Euclidean 2d geometries with two boundaries, without any particular restriction on the boundaries’ mutual position. This amplitude was first calculated using matrix model techniques (for cylinder topology) [36].

To appreciate the underlying construction, consider a given geometry of cylindrical topology. Its two boundaries will be separated by a geodesic distance t , in the sense of minimal distance of any point on the final loop to the initial loop. It follows that we can consider the geometry as composed of a cylinder where the entire final loop (i.e. *each* of its points) has a distance t from the initial one and a “cap” related to the disc amplitude, as illustrated in figure 4(a). One can now obtain the loop-loop amplitude by integrating over all t and all gluings of the cap (we refer to [30] for details). An intriguing aspect of the construction is that the decomposition of a given geometry into cylinders and caps is not unique. One can choose another decomposition consisting of two cylinders of length t_1 and t_2 , with $t_1 + t_2 = t$, joined by a cap, as illustrated in figure 4(b). As shown in [30], the end result is indeed independent of this decomposition.

The whole construction can be repeated for our new, generalized CDT model, in

this way *defining* a loop-loop amplitude. More precisely, although an exact equality of amplitudes corresponding to different decompositions like those depicted in figure 4(a) and (b) is not immediately obvious at the level of the triangulations of the discretized theory,⁷ the continuum ansatz (4.1) below is self-consistent, in the sense that it leads to a non-trivial symmetric expression for the amplitude with a well-defined $g_s \rightarrow 0$ limit. The algebra is similar to that of [30].

We will denote the loop-loop amplitude by $G_{\lambda,g_s}(x, y)$, and its Laplace transform by $G_{\lambda,g_s}(l_1, l_2)$, related in the same way as was discussed for the loop-loop propagator (c.f. eq. (2.6) and the discussion leading up to it). The integral equation corresponding to figure 4 is given by

$$G_{\lambda,g_s}(l_1, l_2) = \int_0^\infty dt \int_0^\infty dl G_{\lambda,g_s}(l_1, l; t) l W_{\lambda,g_s}(l + l_2). \quad (4.1)$$

Laplace-transforming eq. (4.1), the integrals can be performed using eqs. (3.8)-(3.11). After some non-trivial algebra one obtains

$$G_{\lambda,g_s}(x, y) = \frac{1}{f(x)f(y)} \frac{1}{4g_s} \left(\frac{[(x + \alpha) + (y + \alpha)]^2}{(f(x) + f(y))^2} - 1 \right), \quad (4.2)$$

where we are using the notation

$$f(x) = \sqrt{(x + \alpha)^2 - 2g_s/\alpha} = \hat{W}_{\lambda,g_s}(x)/(x - \alpha). \quad (4.3)$$

In the limit $g_s \rightarrow 0$ one finds

$$G_{\lambda,g_s}^{(0)}(x, y) = \frac{1}{2\sqrt{\lambda}(x + \sqrt{\lambda})^2(y + \sqrt{\lambda})^2}, \quad (4.4)$$

a result which could of course also have been obtained directly from (4.1) using (2.4), (2.5) and (3.1). We note that the corresponding expression in the case of Euclidean 2d quantum gravity is given by

$$G_\lambda^{(e)}(x, y) = \frac{1}{2h(x)h(y)(h(x) + h(y))^2}, \quad h(x) = \sqrt{x + \sqrt{\lambda}}, \quad (4.5)$$

which can be obtained from expressions similar to (3.8)-(3.11), only with $\hat{W}_{\lambda,g_s}(x)$ replaced by the Euclidean disc amplitude

$$W_\lambda^{(e)}(x) = (x - \sqrt{\lambda}/2) h(x). \quad (4.6)$$

We observe a structural similarity between (4.2) and (4.6), with the function $f(x)$ having the same relation to $\hat{W}_{\lambda,g_s}(x)$ as $h(x)$ has to $W_\lambda^{(e)}(x)$. The existence of well-defined, symmetric expressions for the unrestricted loop-loop amplitudes in our generalized CDT model (at genus 0) and thus in standard two-dimensional CDT, formulas (4.2) and (4.4), gives strong support to the claims that (i) the proper-time propagator does indeed encode the complete information on the quantum-gravitational system, and (ii) following the arguments given in [30] concerning the decomposition invariance of the loop-loop amplitude (c.f. figure 4), the continuum theory is diffeomorphism-invariant.

⁷because of the different arrangements of the proper-time slicings

5. Discussion

The generalized CDT model of 2d quantum gravity we have defined in this paper is a perturbative deformation of the original model in the sense that it has a convergent power expansion of the form

$$W_{\lambda,g_s}(x) = \sum_{n=0}^{\infty} c_n(x, \lambda) \left(\frac{g_s}{\lambda^{3/2}} \right)^n \quad (5.1)$$

in the dimensionless coupling constant $g_s/\lambda^{3/2}$. This implies in particular that the average number $\langle n \rangle$ of “causality violations” in a two-dimensional universe described by this model is finite, a property already observed in previous 2d models with topology change [20–22]. The expectation value of the number n of branchings can be computed according to

$$\langle n \rangle = \frac{g_s}{W_{\lambda,g_s}(x)} \frac{dW_{\lambda,g_s}(x)}{dg_s}, \quad (5.2)$$

which is finite as long as we are in the range of convergence of $W_{\lambda,g_s}(x)$. As already mentioned, this coincides precisely with the range where the function $W_{\lambda,g_s}(x)$ behaves in a physically acceptable way, namely, $W_{\lambda,g_s}(l)$ goes to zero like exponentially in terms of the length l of the boundary loop. The same is true for the other functions considered, namely, $G_{\lambda,g_s}(l_1, l_2; t)$ and $G_{\lambda,g_s}(l_1, l_2)$.

The behaviour (5.2) should be contrasted with that in 2d Euclidean quantum gravity, and is reflected in the different scaling behaviours (2.10) and (2.12) for the time t . These scaling relations show that the effective continuum “time unit” in Euclidean quantum gravity is much longer than in CDT, giving rise to infinitely many causality violations for a typical space-time history which appears in the path integral when the cut-off a is taken to zero. This phenomenon was discovered in the seminal paper [29].

As we have already mentioned in the introduction, the calculations presented here should be seen as pertaining to the genus-0 sector of a generalized CDT model, which also includes a sum over space-time topologies. Although we have not given a precise definition of the higher-genus amplitudes in this paper, one would expect them to be finite order by order. If the handles are as scarce as are the baby universes in the genus-0 amplitudes, it might even be that the sum over all genera is uniquely defined. Whether or not this is so will clearly also depend on the combinatorics of allowed handle configurations.

In the context of higher-genus amplitudes, it is natural to associate each handle with a “string coupling constant”, because one may think of it as a process where (one-dimensional) space splits and joins again, albeit as a function of an intrinsic proper time, rather than the time of any embedding space. An explicit calculation reveals that in the generalized CDT model this process is related with a coupling constant g_s^2 [37], which one may think of as two separate factors of g_s , associated with the splitting and joining respectively.

How does the disc amplitude fit into this picture? From a purely Euclidean point of view all graphs appearing in figure 3 have the fixed topology of a disc. However, from a Lorentzian point of view, which comes with a notion of time, it is clear that the branching of a baby universe is associated with a change of the *spatial* topology, a singular process

in a Lorentzian space-time [38]. One way of keeping track of this in a Wick-rotated, Euclidean picture is as follows. Since each time a baby universe branches off it also has to end somewhere, we may think of marking the resulting “tip” with a puncture. From a gravitational viewpoint, each new puncture corresponds to a topology change and receives a weight $1/G_N$, where G_N is Newton’s constant, because it will lead to a change by precisely this amount in the two-dimensional (Euclidean) Einstein-Hilbert action

$$S_{EH} = -\frac{1}{2\pi G_N} \int d^2\xi \sqrt{g} R. \tag{5.3}$$

Identifying the dimensionless coupling constant in eq. (2.13) with $g(a) = e^{-1/G_N(a)}$, one can introduce a *renormalized* gravitational coupling constant by

$$\frac{1}{G_N^{\text{ren}}} = \frac{1}{G_N(a)} + \frac{3}{2} \ln \lambda a^2. \tag{5.4}$$

This implies that the *bare* gravitational coupling constant $G_N(a)$ goes to zero like $1/|\ln a^3|$ when the cut-off vanishes, $a \rightarrow 0$, in such a way that the product $e^{1/G_N^{\text{ren}}}/\lambda^{3/2}$ is independent of the cut-off a . We can now identify

$$e^{-1/G_N^{\text{ren}}} = g_s/\lambda^{3/2} \tag{5.5}$$

as the genuine coupling parameter in which we expand.

This renormalization of the gravitational (or string) coupling constant is reminiscent of the famous double-scaling limit in non-critical string theory.⁸ In that case one also has $g_s \propto e^{-1/G_N^{\text{ren}}}$, the only difference being that relation (5.4) is changed to

$$\frac{1}{G_N^{\text{ren}}} = \frac{1}{G_N(a)} + \frac{5}{4} \ln \lambda a^2, \tag{5.6}$$

whence the partition function of non-critical string theory appears precisely as a function of the dimensionless coupling constant $g_s/\lambda^{5/4}$.

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⁸It is called the double-scaling limit since from the point of view of the discretized theory it involves a simultaneous renormalization of the cosmological constant λ and the gravitational coupling constant G_N . In this article we have already performed the renormalization of the cosmological constant. For details on this in the context of CDT we refer to [5].

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